Universidad Simón Bolívar.
Departamento de Matemáticas
Puras y Aplicadas.

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Matemáticas II (MA–1112)

3^er Parcial. (35 %)
TIPO B

1. (3 puntos) Demuestre que $\cosh(2x) = \cosh^2(x) + \operatorname{senh}^2(x)$.

$$\cosh^2(x) + \operatorname{senh}^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4} =$$

$$\frac{2e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} = \cosh(2x)$$

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2. (10 puntos) Calcule las siguientes integrales:

a) Hacemos el cambio de variables:
$x = \sqrt{u}$,
$dx = \dfrac{du}{2\sqrt{u}}$.

$$\int \frac{\ln(\cosh(\sqrt{u}))\tanh(\sqrt{u})}{\sqrt{u}}\, du = 2\int \ln(\cosh(x))\tanh(x)\, dx = 2\int \frac{\ln(\cosh(x))\operatorname{senh}(x)}{\cosh(x)}\, dx =$$

Ahora, hacemos el cambio de variables:
$t = \cosh(x)$,
$dt = \operatorname{senh}(x)dx$.

$$2\int \frac{\ln(t)}{t}\, dt = \ln^2(t) + C = \ln^2(\cosh(\sqrt{u})) + C$$

b) $$\frac{x^2+3x+3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{(A+B)x^2 + (B+C)x + (A+C)}{x^3+x^2+x+1}.$$

$$A+B = 1, \quad B+C = 4, \quad A+C = 4 \qquad \Longrightarrow \qquad A = B = \frac{1}{2}, \quad C = \frac{5}{2}$$

$$\int \frac{x^2+3x+3}{(x+1)(x^2+1)}\, dx = \frac{1}{2}\int \frac{1}{x+1}\, dx + \frac{1}{2}\int \frac{x+5}{x^2+1}\, dx = \frac{1}{2}\int \frac{1}{x+1}\, dx + \frac{1}{2}\int \frac{x}{x^2+1} + \frac{1}{2}\int \frac{5}{x^2+1}\, dx.$$

$$= \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) + \frac{5}{2}\arctan(x) + C = \ln\left(\sqrt[4]{(x+1)^2(x^2+1)}\right) + \frac{5}{2}\arctan(x) + C.$$

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3. Calcule los siguientes límites

a) $$\lim_{x\to 3} \frac{1}{x-3} - \frac{1}{\ln(x-2)} = \lim_{x\to 3} \frac{\ln(x-2) - x + 3}{(x-3)\ln(x-2)} \overset{L'H}{=} \lim_{x\to 3} \frac{\frac{1}{x-2} - 1}{(x-3)\frac{1}{x-2} + \ln(x-2)} = \lim_{x\to 3} \frac{1-(x-2)}{x-3+(x-2)\ln(x-2)} =$$

$$\lim_{x\to 3} \frac{3-x}{x-3+(x-2)\ln(x-2)} \overset{L'H}{=} \lim_{x\to 3} \frac{-1}{1+(x-2)\frac{1}{x-2} + \ln(x-2)} = \lim_{x\to 3} \frac{-1}{2+\ln(x-2)} = -\frac{1}{2}$$

b) $\displaystyle\lim_{x\to 0} \left(\frac{\tan(x)}{x}\right)^{\frac{1}{x^2}}$ hacemos $y = \left(\frac{\tan(x)}{x}\right)^{\frac{1}{x^2}} \Longrightarrow \ln(y) = \frac{1}{x^2}\ln(\frac{\tan(x)}{x})$.

$$\lim_{x\to 0} \frac{1}{x^2}\ln\left(\frac{\tan(x)}{x}\right) = \lim_{x\to 0} \frac{\ln\left(\frac{\tan(x)}{x}\right)}{x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{x}{\tan(x)}\frac{x\sec^2(x) - \tan(x)}{x^2}}{2x} = \lim_{x\to 0} \frac{x}{\tan(x)} \lim_{x\to 0} \frac{x\sec^2(x) - \tan(x)}{2x^3} =$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\sec^2(x) + 2x\sec^2(x)\tan(x) - \sec^2(x)}{6x^2} = \lim_{x\to 0} \frac{2x\sec^2(x)\tan(x)}{6x^2} = \lim_{x\to 0}\sec^2(x) \lim_{x\to 0} \frac{\tan(x)}{3x} = \frac{1}{3}$$

por lo tanto,

$$\lim_{x\to 0} \left(\frac{\tan(x)}{x}\right)^{\frac{1}{x^2}} = e^{1/3}.$$

4. (5 puntos) Halle el volumen del sólido generado al girar la región limitada por $y = 4 - x^2$, $x \geq 0$, $y \geq 0$ alrededor de la recta $x = 2$.

$$\int_0^4 \pi(2^2 - \left(2 - \sqrt{4-y}\right)^2)\,dy = \pi \int_0^4 4\sqrt{4-y} - (4-y)\,dy$$

$$= \pi \left(-\frac{8}{3}(4-y)^{\frac{3}{2}} - 4y + \frac{y^2}{2}\right)\Big|_0^4 =$$

$$\pi \left(0 - 16 + \frac{16}{2} + \frac{64}{3} + 0 - 0\right) = \frac{40\pi}{3}.$$

$$\int_0^2 2\pi(2-x)(4-x^2)\,dx = 2\pi \int_0^2 (x^3 - 2x^2 - 4x + 8)\,dx =$$

$$2\pi \left(\frac{x^4}{4} - 2\frac{x^3}{3} - 2x^2 + 8x\right)\Big|_0^2 =$$

$$2\pi(\frac{16}{4} - \frac{16}{3} - 8 + 16) = 18\pi = \frac{40\pi}{3}.$$

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5. (5 puntos) Diga si la integral $\int_0^{\frac{\pi}{2}} \cot(x)\,dx$ converge o diverge. En caso de que sea convergente, halle su valor.

$$\int_0^{\frac{\pi}{2}} \operatorname{cotan}(x)\,dx = \lim_{a\to 0^+} \int_a^{\frac{\pi}{2}} \operatorname{cotan}(x)\,dx = \lim_{a\to 0^+} \left(\ln(\operatorname{sen}(x))\Big|_a^{\frac{\pi}{2}}\right) = \lim_{a\to 0^+} (\ln(\operatorname{sen}(\pi/2)) - \ln(\operatorname{sen}(a)))$$

$= -\lim_{a\to 0^+} \ln(\operatorname{sen}(a)) = \infty$, luego la integral diverge.

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6. (5 puntos) Halle el o los valores de A para que la integral $\int_0^\infty \frac{Ax}{x^2+1}\,dx$ sea convergente.

$$\int_0^\infty \frac{Ax}{x^2+1}\,dx = \lim_{b\to+\infty} A \int_0^b \frac{x}{x^2+1}\,dx = \lim_{b\to+\infty} A \left(\frac{1}{2}\ln(x^2+1)\Big|_0^b\right) = \lim_{b\to+\infty} A \left(\frac{1}{2}\ln(b^2+1)\right)$$

este límite existe solamente si $A = 0$.